2.3: Acceleration-Velocity Models

In Section 1.2 we discussed vertical motion of a mass m near the surface of the earth. If we neglect air resistance, then Newton's second law of motion ($F = ma$) implies that the velocity of the mass satisfies the equation

$$m\frac{dv}{dt} = F_G \tag{1}$$

where $F_G = -mg$ is the (downward-directed) force of gravity (9.8 m/s^2 or 32 ft/s^2).

Exercise 1. Suppose that a crossbow bolt is shot straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 4.9$ (m/s). Find the maximum height of the bolt and the time it will spend aloft.

$\frac{dv}{dt} = -9.8$, $v = -9.8t + 49$

$y = -4.9t^2 + 49t = -4.9t(t-10)$

Max ht $= 122.5$m

Aloft $= 10$s.

One thing that was not taken into consideration in (1) is the effect of air resistance. Adjusting for this we get

$$m\frac{dv}{dt} = F_G + F_R. \tag{2}$$

Newton showed in his *Principia Mathematica* that certain physical assumptions imply that F_R is proportional to the square of the velocity; i.e. $F_R = kv^2$. However, empirical data shows that this is not quite right and that $F_R = kv^p$ for some $1 \le p \le 2$. Let us consider the extreme cases.

Example 1.(p=1) We suppose $F_R = -kv$ for some constant k. Then we can rewrite (2) as

$$m\frac{dv}{dt} = -kv - mg \quad \text{or} \quad \frac{dv}{dt} = -\rho v - g, \tag{3}$$

where $\rho = k/m > 0$. We can easily solve (3) for

$$v(t) = \left(v_0 + \frac{g}{\rho}\right)e^{-\rho t} - \frac{g}{\rho}.$$

What is the **terminal speed** of the object?

$V_t = \lim_{t\to\infty} v(t) = -\frac{g}{\rho}$

Exercise 2. Suppose that a crossbow bolt is shot straight upward with intial velocity $v_0 = 49$ m/s from ground level. But now assume that air resistance is taken into account with $\rho = 0.04$. Use (3) to find the maximum height and time aloft of the bolt. Compare with Exercise 1.

$v(t) = 294\, e^{-t/25} - 245 = 0$ when $t \approx 25 \ln\left(\frac{294}{245}\right) \approx 4.558$

$Y_{max} \approx 108.28$ m

$y(t) \approx 9.411$ s.

Example 2. (p=2) We suppose $F_R = \pm kv^2 = -kv|v|$. The we can rewrite (2) as

$$m\frac{dv}{dt} = \pm kv|v| - mg \quad \text{or} \quad \frac{dv}{dt} = -g - \rho v|v|, \tag{4}$$

where $\rho = k/m > 0$. Considering the two cases separately, we can easily solve to find

$$y(t) = y_0 + \frac{1}{\rho} \ln \left| \frac{\cos(C_1 - t\sqrt{\rho g})}{\cos C_1} \right| \quad \text{(Upward Motion)}$$

or

$$y(t) = y_0 - \frac{1}{\rho} \ln \left| \frac{\cosh(C_2 - t\sqrt{\rho g})}{\cosh C_2} \right|. \quad \text{(Downward Motion)}$$

Homework. 1-11, 19-25 (odd)